

Optimal Public Expenditure with Inefficient Unemployment: Online Appendices

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Online Appendix A: The Model with Land

We derive several results pertaining to the model with land developed in section 2.4. We also calibrate the model to US data; the calibrated model is simulated in section 5.

Household's Problem and Equilibrium

We solve the household's utility-maximization problem and analyze equilibrium dynamics.

The current-value Hamiltonian of the household's problem is

$$\mathcal{H}(t, c(t), l(t)) = \mathcal{U}(c(t), g(t)) + \mathcal{V}(l(t)) + \lambda(t) \{p(t) [1 - u(x(t))] k - p(t) [1 + \tau(x(t))] c(t) - T(t)\}.$$

It has control variable $c(t)$, state variable $l(t)$, and current-value costate variable $\lambda(t)$. The necessary conditions for an interior solution to the maximization problem are $\partial \mathcal{H} / \partial c = 0$, $\partial \mathcal{H} / \partial l = \delta \lambda(t) - \dot{\lambda}(t)$, and the appropriate transversality condition (see Acemoglu 2009, theorem 7.13). The conditions $\partial \mathcal{H} / \partial c = 0$ and $\partial \mathcal{H} / \partial l = \delta \lambda(t) - \dot{\lambda}(t)$ yield (11) and (12).

Since all the equilibrium variables can be recovered from the costate variable $\lambda(t)$, the equilibrium can be represented as a dynamical system of dimension one, with variable $\lambda(t)$. The variable $\lambda(t)$ satisfies the differential equation $\dot{\lambda}(t) = \delta \lambda(t) - \mathcal{V}'(l_0)$. The steady-state value of $\lambda(t)$ is $\lambda = \mathcal{V}'(l_0) / \delta > 0$. Since $\delta > 0$, the steady state is a source. And since $\lambda(t)$ is a nonpredetermined variable, the equilibrium jumps to its steady-state position at $t = 0$.

As a consequence, in equilibrium, the state variable is constant at $l(t) = l_0$ and the costate variable is constant at $\lambda(t) = \mathcal{V}'(l_0) / \delta$. Since \mathcal{V} is strictly concave, we conclude that the equilibrium path of $c(t)$ and $l(t)$ is in fact the unique global maximum of the household's problem (see Acemoglu 2009, theorem 7.14).

Utility Function

We compute the derivatives of the utility function, given by (8). We use the derivatives to calculate private demand (14). We will also use the derivatives to compute the unemployment multipliers and to calibrate and simulate the model. We first compute first derivatives:

$$\begin{aligned} \frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} &= (1 - \gamma)^{\frac{1}{\epsilon}} \left(\frac{c}{\mathcal{U}} \right)^{\frac{\epsilon-1}{\epsilon}}, & \mathcal{U}_c &\equiv \frac{\partial \mathcal{U}}{\partial c} = \left[(1 - \gamma) \frac{\mathcal{U}}{c} \right]^{\frac{1}{\epsilon}} \\ \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} &= \gamma^{\frac{1}{\epsilon}} \left(\frac{g}{\mathcal{U}} \right)^{\frac{\epsilon-1}{\epsilon}}, & \mathcal{U}_g &\equiv \frac{\partial \mathcal{U}}{\partial g} = \left(\gamma \frac{\mathcal{U}}{g} \right)^{\frac{1}{\epsilon}} \end{aligned}$$

Next, we compute second derivatives:

$$\begin{aligned}\frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} &= \frac{1}{\epsilon} \cdot \left(\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} - 1 \right) \\ \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} &= \frac{1}{\epsilon} \cdot \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)}.\end{aligned}$$

When the Samuelson rule holds, $MRS_{gc} = \mathcal{U}_g/\mathcal{U}_c = 1$, so

$$(A1) \quad (g/c)^* = \frac{\gamma}{1-\gamma}, \quad (g/y)^* = \gamma, \quad (c/y)^* = 1 - \gamma.$$

Hence, at the Samuelson rule, the derivatives simplify to

$$(A2) \quad \begin{aligned}\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} &= 1 - \gamma, & \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} &= \gamma \\ \mathcal{U}_c &= 1, & \mathcal{U}_g &= 1\end{aligned}$$

$$(A3) \quad \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} = -\frac{\gamma}{\epsilon}, \quad \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} = \frac{\gamma}{\epsilon}.$$

Unemployment Multipliers

We compute the unemployment multiplier m , defined by (7), and the empirical unemployment multiplier M , defined by (25). In particular, we establish (16). The multipliers and some of the intermediate results will also be helpful to simulate the model.

First, we compute the effect of public consumption on the price of services. The price is given by (15), which can be written $p(g) = p_0 \mathcal{U}_c(y^* - g, g)^{1-r}$. The elasticity of the price with respect to public consumption therefore is

$$(A4) \quad \frac{d \ln(p)}{d \ln(g)} = (1 - r) \cdot \left[\frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} - \frac{g}{y^* - g} \cdot \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} \right].$$

When unemployment is efficient and public expenditure is at the Samuelson level, the elasticities of \mathcal{U}_c are given by (A3), so we obtain

$$(A5) \quad \frac{d \ln(p)}{d \ln(g)} = (1 - r) \cdot \frac{1}{\epsilon} \cdot \frac{\gamma}{1 - \gamma}.$$

Second, we compute the effects of public consumption and tightness on private demand. Private demand is implicitly defined by (13), which can be written $\mathcal{U}_c(c, g) = p(g) [1 + \tau(x)] \mathcal{V}'(l_0)/\delta$.

The elasticities of private demand with respect to public consumption and tightness therefore are

$$(A6) \quad \frac{\partial \ln(c)}{\partial \ln(x)} = \frac{\eta\tau(x)}{\partial \ln(\mathcal{U}_c)/\partial \ln(c)}$$

$$(A7) \quad \frac{\partial \ln(c)}{\partial \ln(g)} = \frac{\partial \ln(p)/\partial \ln(g) - \partial \ln(\mathcal{U}_c)/\partial \ln(g)}{\partial \ln(\mathcal{U}_c)/\partial \ln(c)}.$$

When unemployment is efficient and public expenditure is at the Samuelson level, we can use (5), (A3), and (A5). Thus, the elasticities of private demand are

$$(A8) \quad \frac{\partial \ln(c)}{\partial \ln(x)} = -(1 - \eta) u^* \frac{\epsilon}{\gamma} \quad \text{and} \quad \frac{\partial \ln(c)}{\partial \ln(g)} = \frac{r - \gamma}{1 - \gamma}.$$

Next, we determine the effect of public consumption on equilibrium tightness. The equilibrium condition determining tightness is (6): $y(x, k) = g + c(x, p(g), g)$. Differentiating this equation with respect to g , we obtain the elasticity of tightness with respect to public consumption:

$$\frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} = \frac{g}{y} + \frac{c}{y} \cdot \left[\frac{\partial \ln(c)}{\partial \ln(g)} + \frac{\partial \ln(c)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} \right]$$

so that

$$(A9) \quad \frac{d \ln(x)}{d \ln(g)} = \frac{(g/y) + (c/y) (\partial \ln(c)/\partial \ln(g))}{\partial \ln(y)/\partial \ln(x) - (c/y) (\partial \ln(c)/\partial \ln(x))}.$$

(In the differentiation, we have assumed that k is fixed; this assumption holds both in section 2.4 and in the simulations.) When unemployment is efficient and public expenditure is at the Samuelson level, we can use (A8), (A1), and $\partial \ln(y)/\partial \ln(x) = 0$. Hence, the elasticity of tightness with respect to public consumption is

$$(A10) \quad \frac{d \ln(x)}{d \ln(g)} = \frac{1}{(1 - \eta) u^*} \cdot \frac{r}{\epsilon} \cdot \frac{\gamma}{1 - \gamma}.$$

Finally, we can compute the unemployment multipliers m and M . Equations (21) and (26) imply that m and M are given by

$$(A11) \quad m = (1 - \eta) \cdot (1 - u) \cdot u \cdot \frac{y}{g} \cdot \frac{d \ln(x)}{d \ln(g)} \quad \text{and} \quad M = \frac{m}{1 - u + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{u} \cdot m}.$$

Combining (A11) with (A10) and (A1), we obtain the values of m and M when unemployment

Table A1. Parameter Values in Simulations

	Description	Source
$\epsilon = 1$	Elasticity of substitution between g and c	Amano and Wirjanto (1997, 1998)
$\gamma = 0.16$	Parameter of utility function	Matches $(G/C)^* = 19.7\%$
$s = 2.8\%$	Monthly separation rate	Landais, Michaillat, and Saez (2018)
$\eta = 0.6$	Matching elasticity	Landais, Michaillat, and Saez (2018)
$\omega = 0.60$	Matching efficacy	Landais, Michaillat, and Saez (2018)
$\rho = 1.4$	Matching cost	Matches $u^* = 6.1\%$
$r = 0.46$	Price rigidity	Matches $M = 0.5$ at $\alpha = 1$
$p_0 = 0.96$	Price level	Matches $u = u^*$ at $\alpha = 1$

is efficient and public expenditure is at the Samuelson level:

$$m = \frac{(1 - u^*)r}{(1 - \gamma)\epsilon} \quad \text{and} \quad M = \frac{r}{\gamma r + (1 - \gamma)\epsilon}.$$

Calibration

We calibrate the model using evidence from the United States. The calibration is summarized in table A1. In section 5, we simulate the calibrated model over the business cycle.

We begin by calibrating the utility function (8). We set the elasticity of substitution between public and private consumption to a plausible midrange estimate: $\epsilon = 1$ (section 4). The utility function is therefore Cobb-Douglas:

$$\mathcal{U}(c, g) = \frac{c^{1-\gamma} g^\gamma}{(1 - \gamma)^{1-\gamma} \gamma^\gamma}.$$

Next, we assume that Samuelson spending is the average level of public expenditure in the United States for 1990–2014: $(G/C)^* = 19.7\%$ (section 4). Since (A1) implies that $\gamma = (G/C)^*/[1 + (G/C)^*]$, we set $\gamma = 0.16$.

We then calibrate matching parameters. The calibration relies on the descriptive statistics provided by Landais, Michaillat, and Saez (2018) for the US labor market between 1990 and 2014. They find a separation rate of $s = 2.8\%$ (online appendix B), a matching elasticity of $\eta = 0.6$ (online appendix D), and a matching efficacy of $\omega = 0.60$ (online appendix G). We use these values. They also find average unemployment rate and tightness of $u = 6.1\%$ and $x = 0.43$ (online appendix G). We assume that these averages are efficient: $u^* = 6.1\%$ and $x^* = 0.43$. Then, to set the matching cost, we use (3), which implies $\rho = \omega x^{-\eta} \tau / [(1 + \tau)s]$. This relation holds for any τ and x , in particular when tightness is efficient. But when tightness is efficient,

$\tau^* = (1 - \eta)u^*/\eta$, so $\tau^* = 4.1\%$. Plugging $x^* = 0.43$ and $\tau^* = 4.1\%$ in the expression for ρ yields $\rho = 1.4$.

Last, we calibrate the price mechanism (15), which can be written $p(g) = p_0 \mathcal{U}_c(y^* - g, g)^{1-r}$. On average in the United States the unemployment multiplier is $M = 0.5$ (section 4). Since we assume that on average unemployment is efficient and the Samuelson rule holds, M satisfies (16); hence, to match $M = 0.5$, we set $r = 0.46$. Finally, we calibrate the price level such that when the demand parameter $\alpha \equiv \delta/\mathcal{V}'(l_0) = 1$, unemployment is indeed efficient. We also assume that the Samuelson rule holds when $\alpha = 1$. We infer from (A2) that when unemployment is efficient and the Samuelson rule holds, (13) becomes $1 = (1 + \tau^*)p_0\alpha$. This condition must be satisfied when $\alpha = 1$; as $\tau^* = 4.1\%$, we need to set $p_0 = 0.96$.

Online Appendix B: Other Examples of Demand Side

In section 2.4 we describe a demand side with land. Here we present two other examples of demand side: one with money and another one with government bonds. We find that they both yield an equilibrium with the same properties as the land equilibrium.

Money in the Utility Function

We replace land by money and assume that households derive utility from real money balances. Introducing money in the utility function is a classical way to generate an aggregate demand: following Sidrauski (1967), a large number of business-cycle models with money in the utility function have been developed (for example, Barro and Grossman 1971; Blanchard and Kiyotaki 1987). The presence of money in the utility function is meant to capture the transaction services provided by money.

The representative household holds $D(t)$ units of money. The supply of money is fixed at D_0 . In equilibrium, the money market clears: $D(t) = D_0$. The price of services in terms of money is $p(t)$. We specify a mechanism for the price of services: $p(t) = p(g(t))$. Let $d(t) \equiv D(t)/p(t)$ be the household's real money balances. The household's instantaneous utility function is $\mathcal{U}(c(t), g(t)) + \mathcal{V}(d(t))$. The law of motion of the household's real money balances is

$$\dot{d}(t) = [1 - u(x(t))]k - [1 + \tau(x(t))]c(t) - \pi(t)d(t) - \frac{T(t)}{p(t)},$$

where $\pi(t) \equiv \dot{p}(t)/p(t)$ is the inflation rate. Since the government maintains public consumption at a constant level g , the price is also constant at $p = p(g)$, and inflation is zero. Accordingly, the household's real money balances follow

$$\dot{d}(t) = [1 - u(x(t))]k - [1 + \tau(x(t))]c(t) - \frac{T(t)}{p},$$

and in equilibrium the household's real money balances are fixed at D_0/p .

The household's problem has the same structure as in the model with land. Hence, as in the model with land, the equilibrium immediately converges to steady state. Private demand $c(x, p, g)$ is implicitly defined by

$$\frac{\partial \mathcal{U}}{\partial c} = [1 + \tau(x)] \frac{\mathcal{V}'(D_0/p)}{\delta},$$

which is almost the same expression as in the model with land. The only difference is that the price of service p affects private demand through a different channel. With land, p is the price

of services relative to land, so it affects private demand through a substitution effect. Here, the price of services relative to real money is 1, but p determines the amount of real money held by households (D_0/p), so it affects private demand through an income effect.

Bonds in the Utility Function

We replace land by government bonds and assume that households derive utility from real bond holdings. Assuming that bonds enter the utility function is a simple way to generate an aggregate demand in a dynamic cashless economy. Several papers in macroeconomics and finance make this assumption (for example, Poterba and Rotemberg 1987; Krishnamurthy and Vissing-Jorgensen 2012; Fisher 2015; Campbell et al. 2017; Del Negro et al. 2017; Michaillat and Saez 2018). Compared to other assets, government bonds have special features: they are particularly safe and liquid (Krishnamurthy and Vissing-Jorgensen 2012); they are also useful to satisfy legal requirements or for “window dressing” (Fair and Malkiel 1971, sec. 2). The presence of bonds in the utility function is meant to capture these features.

The price of services in terms of money is $p(t)$ (here money is only a unit of account). The inflation rate is $\pi(t) = \dot{p}(t)/p(t)$. The inflation rate is determined by a price mechanism: $\pi(t) = \pi(g(t))$. Given the inflation rate, the price of services moves according to $\dot{p}(t) = \pi(t)p(t)$. The initial price $p(0)$ is given.

The representative household holds $B(t)$ bonds. Bonds are in zero net supply. In equilibrium, the bond market clears: $B(t) = 0$. The rate of return on bonds is the nominal interest rate $i(t)$. The nominal interest rate is determined by the central bank. We assume that the central bank potentially responds to inflation and fiscal policy; therefore, it sets an interest rate $i(t) = i(\pi(t), g(t))$.

In the economy there are two goods—services and bonds—and hence one relative price (public and private services have the same price). The price of bonds relative to services is determined by the real interest rate, $r(t) = i(t) - \pi(t)$. Given monetary policy and the price mechanism, the real rate can be written as a function of public consumption: $r(t) = r(g(t)) \equiv i(\pi(g(t)), g(t)) - \pi(g(t))$. We assume that $r(t) < \delta$.

Let $b(t) \equiv B(t)/p(t)$ be the household’s real bond holdings. The household’s instantaneous utility function is $\mathcal{U}(c(t), g(t)) + \mathcal{V}(b(t))$, and the law of motion of its real bond holdings is

$$\dot{b}(t) = [1 - u(x(t))]k - [1 + \tau(x(t))]c(t) + r(t)b(t) - \frac{T(t)}{p(t)}.$$

In equilibrium, the household’s real bond holdings are fixed at 0.

Since the government maintains public consumption at a constant level g , the real interest rate is also constant at $r = r(g)$. Consequently, the household’s problem has the same structure as in

the model with land. Hence, the equilibrium immediately converges to steady state. Furthermore, private demand $c(x, r, g)$ is implicitly defined by

$$\frac{\partial \mathcal{U}}{\partial c} = [1 + \tau(x)] \frac{\mathcal{V}'(0)}{\delta - r}.$$

This expression is almost the same as in the model with land; the difference is that real interest rate r appears instead of the price of services p . This is because the price of services relative to real bonds is determined by r , not by p .

Online Appendix C: Proofs

Proof of Lemma 3

Since MRS_{gc} is a function of g/c , the first-order Taylor expansion of MRS_{gc} at $(g/c)^*$ is

$$(A12) \quad MRS_{gc}(g/c) = MRS_{gc}((g/c)^*) + \frac{dMRS_{gc}}{dg/c} \cdot (g/c - (g/c)^*) + O([g/c - (g/c)^*]^2).$$

In addition, $MRS_{gc}((g/c)^*) = 1$ and

$$\frac{dMRS_{gc}}{d(g/c)} = -\frac{1}{\epsilon} \cdot \frac{1}{(g/c)^*}.$$

Hence, (A12) becomes

$$(A13) \quad 1 - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O([g/c - (g/c)^*]^2).$$

The $1/\epsilon$ in the Taylor expansion is evaluated at $(g/c)^*$. But we can replace it by $1/\epsilon$ evaluated at g/c because the difference between the two is proportional to $g/c - (g/c)^*$. So once the difference is multiplied by $g/c - (g/c)^*$ in (A13), it is absorbed by the term $O([g/c - (g/c)^*]^2)$. Thus, (A13) yields (19).

Next, we write $\partial \ln(y)/\partial \ln(x)$ as a function of u :

$$\frac{\partial \ln(y)}{\partial \ln(x)} = (1 - \eta)u - \eta\tau(u).$$

The function $\tau(u)$ is defined by $\tau(u) = \tau(x(u))$, where $\tau(x)$ is given by (3) and $x(u) = u^{-1}(u)$ is the inverse of the function $u(x)$ given by (2). We have

$$\tau'(u) = \tau'(x) \cdot x'(u) = \frac{\tau'(x)}{u'(x(u))} = \frac{(1 + \tau)\eta\tau/x}{-(1 - \eta)(1 - u)u/x} = -\frac{(1 + \tau)\eta\tau}{(1 - \eta)(1 - u)u}.$$

Equation (5) says that $\eta\tau(u^*) = (1 - \eta)u^*$, which implies

$$\tau'(u^*) = -\frac{1 + \tau(u^*)}{1 - u^*}.$$

Using again $\eta\tau(u^*) = (1 - \eta)u^*$, we obtain

$$-\eta\tau'(u^*) = \frac{\eta + \eta\tau(u^*)}{1 - u^*} = \frac{\eta + (1 - \eta)u^*}{1 - u^*} = \eta + \frac{u^*}{1 - u^*}.$$

Hence, the derivative of $\partial \ln(y)/\partial \ln(x)$ with respect to u at u^* is

$$(1 - \eta) - \eta\tau'(u^*) = \frac{1}{1 - u^*}.$$

Furthermore, $\partial \ln(y)/\partial \ln(x) = 0$ at u^* . Thus, a first-order Taylor expansion of $\partial \ln(y)/\partial \ln(x)$ at u^* yields (20).

Finally, since the elasticity of $u(x)$ with respect to x is $-(1 - \eta)(1 - u)$, we find that

$$m = -y \cdot \frac{u}{g} \cdot \frac{d \ln(u)}{d \ln(g)} = \frac{y}{g} (1 - \eta) (1 - u) u \frac{d \ln(x)}{d \ln(g)} = \frac{y}{x} (1 - \eta) (1 - u) u \frac{dx}{dg}.$$

We obtain (21) by rearranging this equation.

Proof of Lemma 4

We start from (18). First, we approximate $1 - MRS_{gc}$ with (19). Next, we rewrite dx/dg with (21) and approximate $\partial y/\partial x$ with (20). These manipulations yield

$$(A14) \quad \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1 - \eta} \cdot \frac{u - u^*}{(1 - u)(1 - u^*)u} + O([g/c - (g/c)^*]^2 + [u - u^*]^2).$$

We can rewrite (A14) as

$$(A15) \quad \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1 - \eta} \cdot \frac{u - u^*}{(1 - u^*)^2 u^*} + O([g/c - (g/c)^*]^2 + [u - u^*]^2).$$

This is because the difference between $1/[(1 - u)(1 - u^*)u]$ and $1/[(1 - u^*)^2 u^*]$ is $O(u - u^*)$. Once this difference is multiplied by $u - u^*$ in (A14), it is absorbed by the term $O([g/c - (g/c)^*]^2 + [u - u^*]^2)$. We obtain (22) from (A15).

Proof of Proposition 1

The economy starts at an equilibrium $[(g/c)^*, u_0]$, where the unemployment rate u_0 is inefficient. Since $u_0 \neq u^*$, the optimal g/c departs from $(g/c)^*$. In (22), the multiplier m and unemployment rate u are functions of g/c , so they respond as g/c moves away from $(g/c)^*$, and we cannot read the optimal g/c off the formula. In this proof, we derive a formula giving the optimal g/c as a function of fixed quantities.

First, we express the equilibrium values of all variables as functions of $[u, g/c]$. The proof of lemma 3 showed that x and τ can be written as functions of u . Since $y = (1 - u)k/(1 + \tau)$, we can also write y as a function of u . Since $g = y \cdot (g/c)/[1 + g/c]$, g can be written as a function of u

and g/c . As $c = y - g$, c can also be written as a function of u and g/c . Last, since $C = (1 + \tau)c$, $G = (1 + \tau)g$, and $Y = (1 + \tau)y$, we can write C , G , and Y as functions of u and g/c .

Among all pairs $[u, g/c]$, the only pairs describing an equilibrium are those consistent with the equilibrium condition $u = u(x(g))$, where g is the function of u and g/c described above, $x(g)$ is the function defined by (6), and $u(x)$ is the function defined by (2). This equilibrium condition defines the unemployment rate as an implicit function of g/c , denoted $u(g/c)$. Then, the pairs $[u(g/c), g/c]$ for all $g/c > 0$ are the equilibria for all possible levels of public expenditure.

We start by linking u to u_0 and g/c . We write a first-order Taylor expansion of $u(g/c)$ around $u((g/c)^*) = u_0$, subtract u^* on both sides, and divide both sides by u^* :

$$(A16) \quad \frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} + \frac{1}{u^*} \cdot \frac{du}{d \ln(g/c)} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O([g/c - (g/c)^*]^2).$$

To compute $du/d \ln(g/c)$ at $[u_0, (g/c)^*]$, we decompose the derivative:

$$\frac{du}{d \ln(g/c)} = \frac{du}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

First, the definition of the unemployment multiplier implies that

$$\frac{du}{d \ln(g)} = -m \cdot (g/y)^*,$$

where m is evaluated at $[u_0, (g/c)^*]$. Second, we compute $d \ln(g)/d \ln(g/c)$. We have

$$\ln(g/c) = \ln(g) - \ln(y(x(g/c), k) - g).$$

Differentiating with respect to $\ln(g/c)$ yields

$$(A17) \quad 1 = \frac{d \ln(g)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)} + \frac{g}{c} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

Reshuffling the terms, we obtain

$$\frac{d \ln(g)}{d \ln(g/c)} = \frac{c}{y} + \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)}.$$

At u^* , $\partial \ln(y)/\partial \ln(x) = 0$, so at u_0 , $\partial \ln(y)/\partial \ln(x)$ is $O(u_0 - u^*)$. Once this term is multiplied by $g/c - (g/c)^*$ in (A16), it creates a term that is $O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2)$. Thus, we omit

the term $(\partial \ln(y)/\partial \ln(x)) \cdot (d \ln(x)/d \ln(g/c))$ and set

$$\frac{d \ln(g)}{d \ln(g/c)} = (c/y)^*.$$

So far, we have shown that

$$(A18) \quad \frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} - mz_1 \frac{g/c - (g/c)^*}{(g/c)^*} + O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2),$$

where

$$z_1 = \frac{(g/y)^*(c/y)^*}{u^*}.$$

Equation (22) includes a remainder that is $O([u - u^*]^2 + [g/c - (g/c)^*]^2)$. Equation (A18) implies that $(u - u^*)^2$ is $O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2)$. Thus the remainder in formula (22) is $O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2)$. Combining (22) and (A18), we therefore obtain

$$(A19) \quad \frac{g/c - (g/c)^*}{(g/c)^*} = z_0 \epsilon m \left[\frac{u_0 - u^*}{u^*} - mz_1 \frac{g/c - (g/c)^*}{(g/c)^*} \right] + O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2).$$

In (A19), ϵ and m are evaluated at $[u, g/c]$. Instead we can use the values of ϵ and m evaluated at $[u_0, (g/c)^*]$ because the difference between the two values of each statistic is $O([u - u_0] + [g/c - (g/c)^*])$. So once the differences are multiplied by $g/c - (g/c)^*$ and $u_0 - u^*$ in (A19), they are absorbed by $O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2)$. Thus, (A19) yields (23).

To finish the proof, we derive (24). With the previous arguments, (22) can be written

$$\frac{g/c - (g/c)^*}{(g/c)^*} = z_0 \epsilon m \frac{u - u^*}{u^*} + O([u_0 - u^*]^2 + [g/c - (g/c)^*]^2),$$

where ϵ and m are evaluated at $[u_0, (g/c)^*]$. Replacing the left-hand side of this equation by the right-hand side of (23), and dividing everything by $z_0 \epsilon m$, we obtain (24).

Proof of Lemma 5

As $G = [1 + \tau(x(g))]g$ and the elasticity of $1 + \tau(x)$ with respect to x is $\eta\tau$, we have

$$(A20) \quad \frac{d \ln(G)}{d \ln(g)} = 1 + \eta\tau \frac{d \ln(x)}{d \ln(g)} = 1 + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{(1 - u)u} \cdot m,$$

where the last equality is obtained from (21). Furthermore, the definitions of m and M imply

$$m = -y \frac{du}{dg} = -\frac{Y}{1 + \tau(x)} \cdot \frac{du}{dG} \cdot \frac{dG}{dg} = \frac{g}{G} (1 - u) M \frac{dG}{dg} = (1 - u) M \frac{d \ln(G)}{d \ln(g)}.$$

We now plug into this equation the expression for $d \ln(G)/d \ln(g)$ obtained in (A20):

$$m = (1 - u) \cdot M + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{u} \cdot M \cdot m.$$

We obtain (26) by rearranging this equation.

Next, consider a change in public expenditure dG . This change leads to a change du in unemployment and, since $Y = (1 - u)k$, to a change $dY = -k \cdot du$ in output. Hence,

$$\frac{dY}{dG} = -k \frac{du}{dG} = -\frac{Y}{1 - u} \cdot \frac{du}{dG} = M.$$

Online Appendix D: Distortionary Taxation

We introduce endogenous labor supply and a distortionary income tax to study how distortionary taxation affects optimal public expenditure. We compare two approaches to taxation: the traditional approach in public economics and macroeconomics, which uses a linear income tax; and the modern approach in public economics, which uses a nonlinear income tax implemented following the benefit principle. With either approach, the formula for optimal stimulus spending remains the same as when labor supply is fixed. These results are summarized in section 3.3.

Traditional Approach

In the traditional approach to taxation, the government uses a linear income tax τ^L to finance public expenditure. With the linear income tax, the household's labor income becomes $(1 - \tau^L)Y(x, k) = (1 - \tau^L)[1 - u(x)]k$. To finance public expenditure G , the tax rate must be $\tau^L = G/Y = g/y$.

The household chooses k to maximize utility. The marginal rate of substitution between labor and private consumption is $MRS_{kc} = \mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$. As usual, the household supplies labor until the marginal rate of substitution between labor and consumption equals the post-tax real wage:

$$(A21) \quad MRS_{kc} = (1 - \tau^L) \frac{1 - u(x)}{1 + \tau(x)}.$$

Indeed, one unit of labor is only sold with probability $1 - u(x)$. When it is sold, it only yields $1/[1 + \tau(x)]$ units of consumption. Hence, the effective real wage is $[1 - u(x)]/[1 + \tau(x)]$, and the post-tax real wage is $(1 - \tau^L)[1 - u(x)]/[1 + \tau(x)]$.¹

The supply decision is distorted by the income tax: a higher τ^L implies a lower k . In fact, (A21) implicitly defines a function $k(g)$ describing how productive capacity responds to a change in public expenditure and the associated tax change. As the income tax is distortionary, the function $k(g)$ is decreasing in g .

The welfare of an equilibrium is $\mathcal{U}(c, g) - \mathcal{W}(k)$. Given a tightness function $x(g)$ and a capacity function $k(g)$, the government chooses g to maximize $\mathcal{U}(y(x(g), k(g)) - g, g) - \mathcal{W}(k(g))$. The first-order condition of the government's problem is

$$0 = \frac{\partial\mathcal{U}}{\partial g} - \frac{\partial\mathcal{U}}{\partial c} - \mathcal{W}'(k) \frac{dk}{dg} + \frac{\partial\mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} + \frac{\partial\mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

¹Formally, for all the models in section 2 and online appendix B, the first-order condition with respect to k is $\mathcal{W}'(k) = (1 - \tau^L)[1 - u(x)]\lambda$, where λ is the costate variable associated with real wealth in the household's Hamiltonian. We combine this equation and (11) to obtain (A21).

Dividing the condition by $\partial \mathcal{U} / \partial c$, we obtain

$$1 = MRS_{gc} - \left(MRS_{kc} - \frac{\partial y}{\partial k} \right) \cdot \frac{dk}{dg} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Households' optimal labor supply, given by (A21), implies that $MRS_{kc} = (1 - \tau^L)(\partial y / \partial k)$. The government's budget constraint implies that $\tau^L = g/y$. Last, equation (4) implies that $\partial y / \partial k = y/k$. Hence, $-(MRS_{kc} - \partial y / \partial k) = \tau^L y / k = g/k$, and we have proved the following:

LEMMA A1: *With a linear income tax, optimal public expenditure satisfies*

$$(A22) \quad \underbrace{1 - \frac{d \ln(k)}{d \ln(g)}}_{\text{modified Samuelson rule}} = MRS_{gc} + \underbrace{\frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}.$$

Formula (A22) differs from formula (18), but the two have the same structure once the Samuelson rule is modified to account for distortionary taxation. Indeed, formula (A22) can be written as the modified Samuelson rule plus a correction equal to $(\partial y / \partial x) \cdot (dx / dg)$. The statistic $1 - d \ln(k) / d \ln(g) > 1$ in the modified Samuelson rule is the marginal cost of funds; it is more than one because the linear income tax distorts labor supply.

In a situation with distortionary taxation, the Samuelson rule is modified, so we also need to modify the definition of Samuelson spending:

DEFINITION A1: *With a linear income tax, Samuelson spending $(g/c)^*$ is given by the modified Samuelson rule:*

$$MRS_{gc}((g/c)^*) = 1 - \frac{d \ln(k)}{d \ln(g)}.$$

The elasticity $d \ln(k) / d \ln(g) < 0$ is evaluated at optimal public expenditure.

Because the marginal cost of funds $(1 - d \ln(k) / d \ln(g))$ is greater than one, the modified Samuelson rule recommends a lower level of public expenditure than the regular Samuelson rule. Therefore, Samuelson spending is lower with a linear income tax. Nevertheless, since the correction to the Samuelson rule is the same in formula (A22), our sufficient-statistic formula for optimal stimulus spending remains the same:

PROPOSITION A1: *Suppose that the economy is initially at an equilibrium $[(g/c)^*, u_0]$. Then, with a linear income tax, optimal stimulus spending is given by (23) and the unemployment rate under the optimal policy is given by (24), where the statistic z_1 is generalized to allow for*

supply-side responses:

$$z_1 = \frac{(g/y)^*(c/y)^*}{u^*} \cdot \frac{1}{1 - d \ln(k)/d \ln(g)}.$$

The elasticity $d \ln(k)/d \ln(g)$ is evaluated at $[(g/c)^*, u^*]$.

Proof: With a linear income tax, Samuelson spending satisfies

$$MRS_{gc}(g/c^*) = 1 - \frac{d \ln(k)}{d \ln(g)},$$

so formula (A22) implies that optimal public expenditure satisfies

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

As in lemma 3, we have

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*}.$$

Moreover, (20) and (21) remain valid. Combining these results, we obtain (22).

Since formula (22) remains valid, the proof follows the same steps as the proof of proposition 1. The only difference occurs once we reach equation (A17). With a supply-side response to taxation, the equation becomes

$$1 = \frac{d \ln(g)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(k)} \cdot \frac{d \ln(k)}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)} + \frac{g}{c} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

Using the same argument as in the proof of proposition 1, we can omit the term containing the factor $\partial \ln(y)/\partial \ln(x)$. Since $\partial \ln(y)/\partial \ln(k) = 1$, we therefore obtain $d \ln(g)/d \ln(g/c) = (c/y)^*/(1 - d \ln(k)/d \ln(g))$. Using the new expression for $d \ln(g)/d \ln(g/c)$, we conclude the proof just like the proof of proposition 1. \square

The unemployment multiplier in formulas (23) and (24) is a policy elasticity, in the sense of Hendren (2016). It measures the change in unemployment for a change in public expenditure accompanied by the change in taxes maintaining a balanced government budget. In section 3 taxes are not distortionary, so the unemployment multiplier should be measured using a policy reform in which taxes are nondistortionary. Here taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the tax change distorts labor supply.

When taxation is nondistortionary, equation (26) shows that the unemployment multiplier m

in our sufficient-statistic formula is closely related to the empirical unemployment multiplier M . Furthermore, the output multiplier is equal to M , so all our results remain the same if we reformulate them with the output multiplier instead of m . But when taxation is distortionary, things are different, and the output multiplier cannot be used to design optimal public expenditure. With distortionary taxation, (26) remains valid, but the link between the output multiplier and M breaks down. Indeed, output is $Y = (1 - u)k$ so

$$\frac{dY}{dG} = -k \frac{du}{dG} + (1 - u) \frac{dk}{dG} = -\frac{Y}{1 - u} \cdot \frac{du}{dG} + \frac{Y}{k} \cdot \frac{dk}{dG} = M + \frac{Y}{k} \cdot \frac{dk}{dG}.$$

Since taxes are distortionary, $dk/dG < 0$ and

$$M = \frac{dY}{dG} - \frac{Y}{k} \cdot \frac{dk}{dG} > \frac{dY}{dG}.$$

Thus, when a change in taxes distort the capacity supplied by households, the unemployment multiplier M is the output multiplier net of the supply-side response $(Y/k)(dk/dG)$. The supply-side response measures the percent change in labor supply when public expenditure increases by one percent of GDP. As taxation is distortionary, the supply-side response is negative and the unemployment multiplier is larger than the output multiplier. The unemployment multiplier is the correct sufficient statistic whether taxation is distortionary or not. With distortionary taxation, there is a wedge between unemployment and output multipliers equal to the supply-side responses, so the output multiplier is not useful to compute optimal stimulus spending.

Intuitively, an increase in public expenditure affects unemployment and the associated increase in taxes reduces labor supply. The negative effect on labor supply determines the marginal cost of fund and Samuelson spending but has nothing to do with the correction to the Samuelson rule and stimulus spending. The effect on unemployment, on the other hand, determines the correction to the Samuelson rule and stimulus spending. Since the output multiplier conveys information about the effect of public spending on labor supply, it is not directly relevant to stimulus spending. Since the unemployment multiplier measures the effect of public spending on unemployment, it governs optimal stimulus spending.

Modern Approach

We turn to the modern approach to taxation in public economics, which uses a nonlinear income tax implemented according to the benefit principle. The benefit principle, which was introduced by Hylland and Zeckhauser (1979) and fully developed by Kaplow (1996, 1998), is an important result in modern public-economic theory: it states that optimal public expenditure is disconnected

from distortionary taxation.² Hence, extra public expenditure should be financed by a change in the nonlinear tax schedule that leaves all individual utilities unchanged, and thus that does not distort further labor supply.

We assume that the government finances any increase in public expenditure by an increase in nonlinear income tax following the benefit principle: the tax schedule is changed to offset the extra benefit received by any individual from the extra public expenditure. Thus, changing public expenditure does not affect individual utilities and does not alter labor supply.

More precisely, we assume that households choose capacity k to maximize utility, and that public expenditure is funded by a distortionary, nonlinear income tax $T(k)$. We start from an equilibrium $[c, g, x, k]$. To ease notation, we define $\phi(x) = [1 - u(x)] / [1 + \tau(x)]$. With the income tax, the household's disposable income becomes $[1 - u(x)] [k - T(k)]$. In equilibrium, households' disposable income equals their expenses: $[1 - u(x)] [k - T(k)] = [1 + \tau(x)] c$ so $c = \phi(x) [k - T(k)]$.

We implement a small change in public expenditure dg funded by a small tax change $dT(k)$ that satisfies the benefit principle. This change triggers a small change dx in tightness. By the benefit principle, the tax change $dT(k)$ is designed to keep the household's utility constant for any choice of k . For all k , $dT(k)$ satisfies

$$(A23) \quad \mathcal{U}(\phi(x) [k - T(k)], g) = \mathcal{U}(\phi(x + dx) [k - T(k) - dT(k)], g + dg).$$

The left-hand side and right-hand side of the equation define two identical functions of k . This implies that the household does not change his choice of k after the reform: labor supply is unaffected by a change dg funded by the benefit principle.

Taking a first-order expansion of the right-hand side of (A23), and subtracting the left-hand side from the right-hand side, we obtain

$$\frac{\partial \mathcal{U}}{\partial c} \cdot \{\phi'(x) [k - T(k)] dx - \phi(x) dT(k)\} + \frac{\partial \mathcal{U}}{\partial g} \cdot dg = 0.$$

Dividing by $\partial \mathcal{U} / \partial c$ and re-arranging yields

$$\phi'(x) T(k) dx + \phi(x) dT(k) = MRS_{gc} dg + \phi'(x) k dx.$$

Accordingly, the effect of the reform on the government budget balance $R = \phi(x) T(k) - g$ is

$$dR = \phi'(x) T(k) dx + \phi(x) dT(k) - dg = (MRS_{gc} - 1) dg + \frac{\partial y}{\partial x} dx.$$

²See Kaplow (2004) and Kreiner and Verdellin (2012) for a survey of the benefit-principle approach.

(We used $dk = 0$ and $\phi'(x)k = \partial y/\partial x$.) At the optimum, $dR = 0$, so we have proved the following:

LEMMA A2: *Under the benefit principle, optimal public expenditure satisfies (18).*

Under the benefit principle, (18) remains valid and capacity k is not affected by changes in public expenditure. Thus, our sufficient-statistic formula remains valid:

PROPOSITION A2: *Suppose that the economy is initially at an equilibrium $[(g/c)^*, u_0]$. Then, under the benefit principle, optimal stimulus spending is given by (23) and the unemployment rate under the optimal policy is given by (24).*

Under the benefit principle, although taxation is distortionary, we obtain the same results as with a fixed labor supply. Furthermore, since there are no labor-supply distortions for a marginal increase in public expenditure, output and unemployment multipliers are equal, and the output multiplier can be used to design optimal stimulus spending.

Online Appendix E: Fixprice Model

We compute the amount of stimulus spending required to fill the output gap in the fixprice model developed in section 3.4. In addition, we present an extension of the fixprice model in which productive capacity is endogenous, not fixed. We derive a sufficient-statistic formula for optimal public expenditure in that extended model.

Stimulus Spending Required to Fill the Output Gap

We derive (29). The economy starts at an equilibrium $[(g/c)^*, y_0]$, where output $y_0 < k$ is inefficiently low. We compute the stimulus spending $g/c - (g/c)^*$ required to fill the output gap $k - y_0$. To that end, we link y to g/c . We write a first-order Taylor expansion of $y(g/c)$ around $y((g/c)^*) = y_0$, evaluate it at $y(g/c) = k$, and divide it by y_0 :

$$(A24) \quad \frac{k - y_0}{y_0} = \frac{d \ln(y)}{d \ln(\ln(g/c))} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O([g/c - (g/c)^*]^2).$$

Next we compute $d \ln(y)/d \ln(g/c)$ using the following decomposition:

$$(A25) \quad \frac{d \ln(y)}{d \ln(g/c)} = \frac{d \ln(y)}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)} = (g/y)^* \cdot \frac{dy}{dg} \cdot \frac{d \ln(g)}{d \ln(g/c)},$$

where dy/dg is evaluated at $[(g/c)^*, y_0]$.

The last step is to compute $d \ln(g)/d \ln(g/c)$. We have $\ln(g/c) = \ln(g) - \ln(y - g)$. Differentiating this equation with respect to $\ln(g/c)$ yields

$$1 = \frac{d \ln(g)}{d \ln(g/c)} - (y/c)^* \frac{d \ln(y)}{d \ln(g/c)} + (g/c)^* \frac{d \ln(g)}{d \ln(g/c)}.$$

Using (A25) and reshuffling the terms, we obtain

$$(A26) \quad \frac{d \ln(g)}{d \ln(g/c)} = \frac{1}{1 + (g/c)^* - (g/c)^*(dy/dg)}.$$

Finally we combine all the results. Plugging (A26) into (A25), we find

$$(A27) \quad \frac{d \ln(y)}{d \ln(g/c)} = \frac{(g/y)^*(dy/dg)}{1 + (g/c)^* - (g/c)^*(dy/dg)} = \frac{(c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)}.$$

Combining (A27) with (A24), we then obtain

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1 - (g/y)^*(dy/dg)}{(c/y)^*(g/y)^*(dy/dg)} \cdot \frac{k - y_0}{y_0} + O([g/c - (g/c)^*]^2),$$

where the output multiplier dy/dg is evaluated at $[(g/c)^*, y_0]$. This equation yields (29).

Endogenous Productive Capacity

We extend the fixprice model by introducing endogenous productive capacity, and we describe optimal public expenditure in that model. We could introduce endogenous capacity by assuming that households are price-takers: they supply capacity k to maximize utility given the price of services. This assumption has a downside, however: it introduces an internal inconsistency in the model when there is excess supply. Indeed, aggregate supply would describe how much households desire to work for a given price, assuming that they can sell all the services that they supply to the market. In reality, households are unable to sell all their services because there is excess supply. To be consistent, the model should allow households to revise their supply decision given that the probability to sell a given service is less than one. But the fixprice model does not allow for this.³

We address this issue as in the New Keynesian literature. We assume that households are price-setters: they set the price of services to maximize profits and supply the amount of services demanded at the profit-maximizing price. When the price is fixed, households simply supply as many services as required to satisfy demand (for example, Nakamura and Steinsson 2014, p. 773). Let y be aggregate output of services, which is demand-determined. Since households supply exactly the amount of services required by demand, aggregate supply of services is $k = y$.

The government now chooses g to maximize $\mathcal{U}(y - g, g) - \mathcal{W}(y)$. The first-order condition of the maximization is

$$(A28) \quad 1 = MRS_{gc} + \frac{dy}{dg} \cdot (1 - MRS_{kc}),$$

where $MRS_{kc} = \mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$ is the marginal rate of substitution between labor and private consumption. This equation is the same as (28), except that the output multiplier is multiplied by the labor wedge $1 - MRS_{kc}$.⁴ This equation is also the same as equation (45) in Woodford (2011)—this is not surprising since our fixprice model has the same ingredients as Woodford's

³The matching model addresses this issue by introducing a matching function that gives the probability to sell services, and by letting households take the probability into account when they make their supply decision.

⁴The labor wedge plays an important role in macroeconomics (see Shimer 2009).

New Keynesian model.

The economy can be in three possible regimes, depending on the labor wedge : efficient production when $1 - MRS_{kc} = 0$, insufficient production when $1 - MRS_{kc} > 0$ (a slump), and excessive production $1 - MRS_{kc} < 0$ (a boom). When there is efficient production, $MRS_{kc} = 1$ and the Samuelson rule remains valid. When there is excessive or insufficient production, things change: $MRS_{kc} \neq 1$ so the correction to the Samuelson rule is nonzero.

We assume that the economy starts at $[(g/c)^*, y_0]$, with a marginal rate of substitution $(MRS_{kc})_0 \neq 1$. Following the procedure developed in the matching model, we obtain a formula expressed as a function of fixed (not endogenous) sufficient statistics:

PROPOSITION A3: *Suppose that the economy is initially at an equilibrium $[(g/c)^*, y_0]$. Then optimal stimulus spending satisfies*

$$(A29) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx \frac{\epsilon \cdot (dy/dg)}{1 + z_3 \epsilon \frac{(dy/dg)^2}{1 - (g/y)^*(dy/dg)}} [1 - (MRS_{kc})_0].$$

The statistics ϵ and dy/dg are evaluated at $[(g/c)^*, y_0]$. Further,

$$z_3 = \frac{(MRS_{kc})_0 (c/y)^*(g/y)^*}{\kappa},$$

where κ is the Frisch elasticity of labor supply:

$$\frac{1}{\kappa} = \frac{d \ln(\mathcal{W}'(k))}{d \ln(k)}.$$

Under the optimal policy, the labor wedge is

$$(A30) \quad 1 - MRS_{kc} \approx \frac{1}{1 + z_3 \epsilon \frac{(dy/dg)^2}{1 - (g/y)^*(dy/dg)}} [1 - (MRS_{kc})_0].$$

The approximations (A29) and (A30) are valid up to a remainder that is $O([g/c - (g/c)^*]^2)$.

Proof: Optimal stimulus spending satisfies (A28), which can be rewritten using (19):

$$(A31) \quad \frac{g/c - (g/c)^*}{(g/c)^*} = \epsilon \cdot \frac{dy}{dg} \cdot (1 - MRS_{kc}) + O([g/c - (g/c)^*]^2).$$

As in the matching model, MRS_{kc} responds to g/c when it deviates from $(g/c)^*$, so we cannot use (A31) to compute optimal stimulus spending. We follow the procedure developed in the matching model and re-express (A31) as a function of fixed sufficient statistics.

To that end, we analyze how MRS_{kc} respond to g/c . In this demand-determined economy, the aggregate-demand relationship always holds. Since the asset (land in our baseline model) is in fixed supply and prices are fixed, the marginal utility of private consumption ($\partial\mathcal{U}/\partial c$) is fixed and does not change when public consumption changes.⁵ Hence, we only consider how the marginal disutility of labor ($\mathcal{W}'(k)$) reacts to public consumption. We find

$$\frac{d \ln(MRS_{kc})}{d \ln(g/c)} = \frac{d \ln(\mathcal{W}'(k))}{d \ln(g/c)} = \frac{1}{\kappa} \cdot \frac{d \ln(y)}{d \ln(g/c)},$$

where κ is the Frisch elasticity of labor supply. Using (A27), we obtain

$$\frac{d \ln(MRS_{kc})}{d \ln(g/c)} = \frac{1}{\kappa} \cdot \frac{(c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)}.$$

Accordingly, the first-order Taylor expansion of $MRS_{kc}(g/c)$ around $(g/c)^*$ is

$$MRS_{kc} = (MRS_{kc})_0 + \frac{1}{\kappa} \cdot \frac{(MRS_{kc})_0 (c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O([g/c - (g/c)^*]^2).$$

In the equation the multiplier dy/dg and elasticity κ are evaluated at $[(g/c)^*, y_0]$. To obtain (A29), we plug this expression for MRS_{kc} into (A31) and reshuffle the terms. Finally, we obtain (A30) by combining (A31) and (A29). \square

Formula (A29) is similar to formula (23) in the matching model; the principal difference is that the amount of inefficiency is measured by the labor wedge $1 - (MRS_{kc})_0$ instead of the unemployment gap. Nonetheless the formula has similar implications. First, with a positive output multiplier, optimal stimulus spending is positive in slumps but negative in booms. Second, optimal stimulus spending is a hump-shaped function of the output multiplier. Third, optimal stimulus spending is larger when public consumption substitutes more easily for private consumption. Last, optimal stimulus spending only partially reduces the output gap: MRS_{kc} is brought closer to one, but remains below one.

Overall, the fixprice model with endogenous capacity leads to similar insights as the matching model. This is reassuring: irrespective of how productive inefficiency is modeled, stimulus spending obeys similar general principles. Compared to the fixprice model with fixed capacity, three differences arise: (a) the model offers a symmetric treatment of excessive production and insufficient production; (b) it is never optimal to completely fill the output gap; and (c) optimal stimulus spending is a smooth function of the sufficient statistics.

⁵For example, in the demand side with land of section 2.4, aggregate demand is given by $\partial\mathcal{U}/\partial c = p\mathcal{V}'(l_0)/\delta$. This relationship always holds since the economy is demand-determined. As l_0 and p are fixed, $\partial\mathcal{U}/\partial c$ does not respond to g .

Yet, for several reasons, the matching model seems more convenient than the fixprice model with endogenous capacity to think about optimal public expenditure. A first limitation of the fixprice model is that its description of booms is not fully satisfactory. When there is excessive production, $MRS_{kc} > 1$ which implies $\mathcal{W}'(k) > \partial\mathcal{U}/\partial c$: people, constrained to supply the amount of services demanded, are working more than they would like. If workers were not bound to supply whatever is demanded, all of them would stop providing services, as the cost of providing each service is higher than the income received. In the matching model, in contrast, all relationships generate surplus for both buyer and seller.

Another limitation of the fixprice model is that the supply side is irrelevant, as the equilibrium is demand-determined. An implication is that distortionary taxation has no effect at all. In contrast, in the matching model, both supply and demand determine the equilibrium, so distortionary taxation reduces output. The matching model is therefore well suited to study the effect of distortionary taxation on optimal public expenditure—something we do in section 3.3.

A last limitation of the fixprice model is that the labor wedge $1 - (MRS_{kc})_0$ is more challenging to measure than the unemployment gap $u_0 - u^*$. As a result, the fixprice formula (A29) is less convenient to apply than the matching formula (23). Indeed, since u_0 is observable, measuring the unemployment gap only requires to measure the efficient unemployment u^* . This can be done from (5), following the method developed by Landais, Michaillat, and Saez (2018). This can also be done by using historical unemployment data, since u^* does not respond to typical macroeconomic shocks and is therefore stable over time (see section 4). In contrast, it is difficult to measure the labor wedge because it is not possible to relate $(MRS_{kc})_0$ to observable variables.⁶ One strategy to measure $(MRS_{kc})_0$ would be to assume that output is efficient before the shocks and that the utility functions \mathcal{W} and \mathcal{U} are stable. Then we could recover $(MRS_{kc})_0$ from the observed change in output, the Frisch elasticity (to link the output change to the change in $\mathcal{W}'(k)$), and a coefficient of risk aversion (to link the output change to the change in $\partial\mathcal{U}/\partial c$). This strategy could work with aggregate-demand shocks but not with aggregate-supply shocks, as the disutility from labor \mathcal{W} varies under such shocks. Hence, it is generally not possible to measure the labor wedge.

⁶For the same reason, it is difficult to measure the New Keynesian output gap in the data (Gali 2008, pp. 80–81).

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