Problem Set on Optimal Control

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Problem 1

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose a consumption path $\{c_t\}_{t\geq 0}$ to maximize utility

$$\int_0^\infty e^{-\rho \cdot t} \cdot \ln(c_t)dt$$

subject to the law of motion of capital

$$\dot{k}_t = f(k_t) - c_t - \delta \cdot k_t.$$

The discount factor $\rho > 0$, and the production function f satisfies

$$f(k) = A \cdot k^\alpha,$$

where $\alpha \in (0, 1)$ and $A > 0$.

A) Write down the present-value Hamiltonian.

B) Show that the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho).$$

C) Solve for the steady state of the system.

Problem 2

Consider the following investment problem: Given initial capital k_0, choose the investment path $\{i_t\}_{t \geq 0}$ to maximize profits

$$\int_0^\infty e^{-r \cdot t} \left[f(k_t) - i_t - \frac{\chi}{2} \cdot \left(\frac{i_t^2}{k_t} \right) \right] dt$$

subject to the law of motion of capital (we assume no capital depreciation)

$$\dot{k}_t = i_t.$$

The interest rate $r > 0$, the capital adjustment cost $\chi > 0$, and the production function f satisfies $f' > 0$ and $f'' < 0$.

A) Write down the current-value Hamiltonian.

B) Use the optimality conditions for the current-value Hamiltonian to derive the following differential equations:

$$\dot{k}_t = \left(\frac{q_t - 1}{\chi} \right) \cdot k_t$$

$$\dot{q}_t = r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi} (q_t - 1)^2$$

C) Solve for the steady state.